Lectures II – III: Reductive Perturbation Theory and Mode-Jumping

Models: - free, forced Duffing
- Nonlinear KG

Key Idea:
- origin of unphysical secularities
- removal by nonlinear frequency shift

Key Consequence

- nonlinear resonance behavior
- mode jumping.

Important examples:

- free Duffing
- NL Klein-Gordon
- forced Duffing.

c) Nonlinear Oscillators – Conservative

→ Here, concerned with $\left\{\begin{matrix}\text{nonlinear}\\ \text{conservative}\end{matrix}\right\}$ oscillator systems, usually small perturbations about/away from the SHO

Prototype: Duffing's Equation

$$\underbrace{\ddot{x} + \omega_0^2 x}_{\text{SHO},\ \omega_0} + \underset{\substack{\downarrow \\ \text{NL, anharmonic term}}}{\epsilon x^3} = 0$$

Observe:

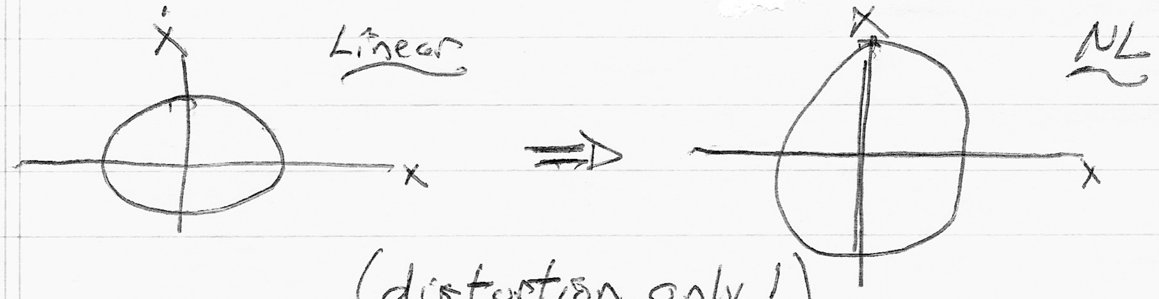
– $H = \frac{1}{2}\dot{x}^2 + \frac{\omega_0^2 x^2}{2} + \epsilon \frac{x^4}{4}$

so $V(x) = \frac{1}{2}\omega_0^2 x^2 + \epsilon \frac{x^4}{4}$ $\qquad (V(x) > 0)$

– natural question to ask re: $\omega = \omega(\epsilon)$?
i.e. evolution of periodic/quasi-periodic orbits upon perturbation.

Now note : V(x) bounded (from below) ⇒ phase space trajectory / contours "contained"

i.e. $(\omega_0 = 1)$



(distortion only!)

so orbit must be bounded!

- A clue;

Observe : $\omega^2 = \frac{1}{2}\omega_0^2\langle x^2\rangle \Big/ \frac{\langle x^2\rangle}{2}$

for SHO. $\left(\langle\ \rangle = \frac{1}{\tau}\int_0^\tau\right)$

⇒

∴ might expect, taking $\tau = 2\pi/\omega_0$

$$\omega^2 \equiv \left(\frac{1}{2}\omega_0^2\langle x^2\rangle + \frac{\epsilon}{4}\langle x^4\rangle\right) \Big/ \frac{\langle x^2\rangle}{2}$$

(solve with SHO 'trial' fctn.)

Now; $\langle x^4\rangle = \langle a^4(\cos\omega_0 t)^4\rangle$

$= a^4\ 3/4)$

$$\boxed{\omega^2 = \omega_0^2 + \frac{3}{4}\epsilon a^2}$$ → amplitude dependent frequency! (NL frequency shift)

i.e. $\omega = \omega_0$ →

$\omega = \omega(\omega_0, \epsilon, \underline{\underline{a}})$! → frequency becomes amplitude dependent.

⟷ (nearly) correct result.

- ▷ Systematics – Calculational Procedure

– expand in ε ! (surprise !?)

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0 \quad ; \quad x(0) = 0$$

$$x = \underline{x^{(0)} + \epsilon x^{(1)}} + \ldots$$

∴ $O(\epsilon^0)$: $\ddot{x}^{(0)} + \omega_0^2 x^{(0)} = 0$

$$x^{(0)} = a\sin(\omega_0 t)$$

start ϵx^{1}

$O(\epsilon^1)$: $\ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\cancel{\epsilon}(x^{(0)})^3$

but $(x^{(0)})^3 = a^3 \sin^3 \omega_0 t$

$$= \frac{a^3}{4}(3\sin\omega_0 t + \sin 3\omega_0 t)$$

start

$$\Rightarrow \quad \ddot{x}^{(1)} + \omega_0^2 x^{(1)} = -\cancel{\epsilon}\frac{a^3}{4}(3\sin\omega_0 t + \sin 3\omega_0 t)$$

↓

{ resonant drive
⇒ secularity !! }

yields $x^{(1)} = \text{homog} + c\,t\cos\omega_0 t$

↓

¿ is this physical ?

$$\Rightarrow \; -\omega_0^2 c t \cos\omega_0 t - 2c\,\omega_0 \sin\omega_0 t$$

$$+\omega_0^2 c t \cos\omega_0 t = -\frac{\epsilon a^3}{4}(3\sin\omega_0 t)$$

$c = \dfrac{3\epsilon}{8\omega_0}$; $\epsilon \sim (\text{freq.})^2$, dimensionally

$$\therefore \; x = a\sin\omega_0 t + \underbrace{\frac{3}{8}\frac{\epsilon}{\omega_0} t \cos\omega_0 t}_{} + \ldots\ldots$$

↓ secularity → $|x|$ diverges linearly in time

→ Unphysical ⇒ recall closed phase space trajectories!

What's going on ?!

→ ω shift

Aside: A trivial example!?!!

Bender Orszag

Kevorkian Cole

Mitropolsky

$$\ddot{x} + (1+\epsilon)^2 x = 0 \; ; \quad x(0) = 0, \;\; \dot{x}(0) = 1$$

if expand in ϵ ;

$$\ddot{x} + x + 2\epsilon x + \epsilon^2 x = 0 \qquad \text{(exactly solvable)}$$

$\epsilon^{(0)}$; $\ddot{x}^{(0)} + x^{(0)} = 0$; $x^{(0)} = \sin t$

$O(\epsilon)$; $\quad \ddot{x}^{(1)} + x^{(1)} = -2\, x^{(0)}$

$$\Rightarrow \quad x^{(1)} = c\, t\left[\cos t\right] + \text{homog:}$$

$$-2c \sin t - ct \cancel{\cos t} + ct \cancel{\cos t} = -2 \sin t$$

$$c = 1$$

. secularity !?!

This is clearly idiotic, since we all know

$x(t) = \sin[(1+\epsilon)t]$ trivially solves the problem!

↳ frequency shift

Moral of this story:

② – to avoid secularity, must allow frequency shift; i.e. here $\omega = 1 \rightarrow \omega = 1+\epsilon$
gives all a warm, fuzzy .....

① – secularity results from breakdown of naive expansion in ϵ at long times, observe:

i.e. $\quad x(t) = \sin[(1+\epsilon)t]$,

Taylor expansion in ϵ $\Rightarrow \quad \cong \sin t + \epsilon t \cos t$

↳ secularity → {artifact of expansion

→ The Fix:

- admit nonlinear frequency shift

(i.e. method of Poincare - Linstedt)
Tip of Reductive P.T. ice-berg ....

- trick is to:

a) expand x, ω on equal footing

$$x = x^{(0)} + \epsilon x^{(1)} + \ldots$$
$$\omega = \omega^{(0)} + \epsilon \omega^{(1)} + \ldots$$

$\ddot{x} + \omega_0^2 (x + \epsilon x^3) = 0$

and use/choose $\omega^{(1)}$, etc. to cancel secularities (renorm.)

i.e. "solvability condition" ⟺ secularity removal.

i.e. $$\frac{d^2x}{dt^2} + x + \epsilon x^3 = 0$$

Scale to ω_0.
$\sigma = \omega_0$.

usual ↓ long time behavior ....

now: $$t = s(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)$$

↑ time param.

and $$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$$

Improved Details

$$\ddot{x} + \omega_0^2 x + \epsilon x^3 = 0$$

now,

$$\left\{ \begin{array}{l} x = x^{(1)} + \underbrace{x^{(2)}}_{O(\epsilon)} + x^{(3)} \rightarrow O(\epsilon^2) \\ \\ \omega = \omega_0 + \omega^{(1)} + \omega^{(2)} + \dots \\ \qquad\quad \rightarrow O(\epsilon) \text{ (shift!)} \end{array} \right.$$

and, $x_0^{(1)} = a \cos \omega t$

$\rightarrow$ note ω! (not ω_0) $\Rightarrow$ note $\omega \to \omega_0$ if shift ignored.

Now, re-write:

$$\underset{\hookrightarrow \text{added}}{\frac{\omega_0^2}{\omega^2}} \ddot{x} + \omega_0^2 x = -\epsilon x^3 - \overset{\text{added}}{\left(1 - \frac{\omega_0^2}{\omega^2}\right)} \ddot{x}$$

added $\frac{\omega_0^2}{\omega^2}$ both sides

extra terms assure LHS = 0, in lowest order.

$\Rightarrow$

$$\frac{\omega_0^2}{\omega^2}\left(-\omega^3 x^{(1)} + \ddot{x}_2\right) + \omega_0^3 \left(x^{(1)} + x_2\right)$$

$$= -\epsilon\left(x^{(1)} + x^{(2)}\right)^3 - \left(1 - \frac{\omega_0^2}{\omega^2}\right)\left(-\omega^2 x^{(1)} + \ddot{x}_2\right)$$

$$\cancel{-\omega_0^2 x^{(1)}} + \ddot{X}_2 + \cancel{\omega_0^3 x^{(1)}} + \omega_0^2 X_2$$

$$= -\epsilon X^{(1)3} - \left(-\omega^2 + \omega_0^2\right) X^{(1)}$$

$$- \left(1 - \frac{\omega_0^2}{\omega^2}\right) \ddot{X}_2$$

$\Rightarrow$

$$\ddot{X}_2 + \omega_0^2 \ddot{X}_2 = \underbrace{-\epsilon X^{(1)3}}_{O(\epsilon)} + \underbrace{(\omega^2 - \omega_0^2)}_{O(\epsilon)} X^{(1)}$$

$$\underset{\substack{\downarrow \\ \text{shift}}}{-} \overset{\text{h.o.}}{\cancel{\left(1 - \frac{\omega_0^2}{\omega^2}\right)}} \ddot{X}_2 \qquad O(\epsilon^2)$$

↳ i.e. $X_2 \sim O(\epsilon)$

$\left(1 - \frac{\omega_0^2}{\omega^2}\right) \sim O(\epsilon)$

$$\omega^2 = \underline{\omega_0^2 + 2\omega_0\omega_1\epsilon} \quad ; \quad \omega^2 - \omega_0^2 = 2\omega_0\omega_1\epsilon$$

$$x^{(1)3} = a^3 \cos^3\omega t = a^3\left[\frac{1}{2} + \frac{\cos 2\omega t}{2}\right]\cos\omega t$$

$$= a^3\left[\frac{\cos\omega t}{2} + \frac{1}{2}\left(\frac{1}{2}\right)\left(\cos 3\omega t + \cos\omega t\right)\right]$$

$$= a^3\left[\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right]$$

$$\ddot{x}_2 + \omega_0^2 x_2 = -\epsilon\left[a^3\left(\frac{3}{4}\cos\omega t + \frac{1}{4}\cos 3\omega t\right)\right]$$
$$+ 2\omega_0\omega_1\epsilon a \cos\omega t$$

Now to dodge the secularity, need cancel all resonant terms on RHS of

$\Rightarrow$

$$\ddot{x}_2 + \omega_0^2 x_2 = -\epsilon\left[\cos\omega t\right]\left[\frac{3}{4}a^3 - 2\omega_0\omega_1 a\right]$$
$$-\epsilon\frac{a^3}{4}\cos 3\omega t$$

$$\therefore \quad \boxed{\omega_1 = \frac{3}{8}a^2/\omega_0}$$

and

$$\boxed{\omega = \omega_0 + \underbrace{\epsilon\left(\frac{3}{8}a^2/\omega_0\right)}_{\text{NL frequency shift!}} + \cdots}$$

and crank ⇒

$$x^{(2)} = -\frac{1}{2}\left(\sigma a^3/16\omega_0^2\right)\cos 3\omega t.$$

Point:

- need get frequencies correct to avoid unphysical resonances, secularities...

- frequency correction ⇒ NL frequency shift.

N.B. : compare:

- exact : $\omega_1 = 3/8\, a^2/\omega_0$

- rough : $\omega_1 = 3/4\, a^2/\omega_0$.

Moral : use frequency shift to eliminate secularity causing term on RHS.

So:

$$\frac{d^2}{ds^2}\left(\overset{\checkmark}{x_0}+\epsilon \overset{\times}{x_1}+\epsilon^2 x_2+\dots\right)+\left[\left(\overset{\checkmark}{x_0}+\epsilon \overset{\times}{x_1}+\epsilon^2 x_2+\dots\right)\right.$$

$$\left.+\epsilon\left(\overset{\times}{x_0}+\epsilon x_1+\epsilon^2 x_2+\dots\right)^3\right]\left(1+\epsilon \overset{\times}{\omega_1}+\epsilon^2\omega_2+\dots\right)^2=0$$

$O(\epsilon^0)$: $$\frac{d^2x_0}{ds^2}+x_0=0$$

$O(\epsilon^1)$: $$\frac{d^2x_1}{ds^2}+x_1+x_0^3+2\omega_1 x_0=0$$

$O(\epsilon^2)$: $$\frac{d^2x_2}{ds^2}+x_2=-3x_0^2x_1-2\omega_1\left(x_1+x_0^3\right)-\left(\omega_1^2+2\omega_2\right)x_0$$

etc.

Now,

$O(\epsilon^0)$: $$\frac{d^2x_0}{ds^2}+x_0=0$$

$$x_0=a\cos\left(s+\overset{\text{phase}}{\phi}\right)$$

$O(\epsilon^1)$:

$$\frac{d^2x_1}{ds^2} + x_1 = -x_0^3 - 2\omega_1 x_0$$

$$= -a^3\cos^3(s+\phi) - 2\omega_1 a\cos(s+\phi)$$

etc

But

$$\cos^3(s+\phi) = \cos(s+\phi)\left[\frac{1}{2} + \frac{1}{2}\cos[2(s+\phi)]\right]$$

$$= \frac{1}{2}\cos(s+\phi) + \frac{1}{2}\cos(s+\phi)\cos[2(s+\phi)]$$

$$= \frac{\cos(s+\phi)}{2} + \frac{1}{4}\cos[3(s+\phi)] + \frac{1}{4}\cos[(s+\phi)]$$

$$= \frac{3}{4}\cos(s+\phi) + \frac{1}{4}\cos(3(s+\phi))$$

$\Rightarrow$

$$\frac{d^2x_1}{ds^2} + x_1 = -a^3\left(\overset{\textcircled{1}}{\frac{3}{4}\cos(s+\phi)} + \overset{\textcircled{2}}{\frac{1}{4}\cos(3(s+\phi))}\right)$$

$$\overset{\textcircled{3}}{-2\omega_1 a\cos[(s+\phi)]}$$

So

①, ③ $\sim \cos(s+\phi)$

resonates with RHS $\Leftrightarrow$ will drive secularity.

② $\sim \cos[3(s+\phi)]$

non-secular drive → harmless

$$\frac{d^2x_1}{ds^2}+x_1=\left[-\frac{3}{4}a^2-2\omega_1\right]a\cos(s+\phi)$$

$$+\frac{1}{4}\cos\left(3(s+\phi)\right)$$

so $\boxed{\omega_1=-\frac{3}{8}a^2 \quad \text{removes secularity!!}}$

$$t=s\left(1-\frac{3}{8}\epsilon a^2+\cdots\right)$$

$$\boxed{\omega=\omega_0\left[1+\frac{3}{8}\epsilon a^2+\cdots\right]}$$

NL frequency shift!

i.e.

$s=t/1-\frac{3}{8}\epsilon a^2$

$\cong t\left(1+\frac{3}{8}\epsilon a^2\right)$

and $x_1=\frac{1}{32}a^3\cos[3(s+\phi)]$

Key Point: ⇒ By expanding ω in ϵ, method of Poincare and Lindstedt introduces additional degrees of freedom, so one can remove secularity order-by-order in P.T.

Pf. of asymptotics

⇒ essence of NL oscillator is NL frequency shift, i.e.

$$\omega = \omega_0 + \frac{3\epsilon}{8\omega_0}a^2$$

) Forced Anharmonic Oscillator – Mode Jumping

- here, consider next step in development

⇒ forced Duffing's eqn.

$$\ddot{x} + \underset{\substack{\downarrow \\ \text{friction}}}{2\lambda\dot{x}} + \omega_0^2 x + \underset{\substack{\downarrow \\ \alpha=0,\text{ initially}}}{\cancel{\alpha x^2}} + \beta x^3 = \frac{f_{ext}}{m}$$

$$\text{issue: combination} \begin{cases} \text{resonance} \\ \text{forcing} \\ \text{NL} \end{cases}$$

→ Frequency shift + recovers guesstimate (almost...)

→ example 2 – Nonlinear Klein Gordon Eqn.
mention

KG eqn:

$$\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t}-\frac{\partial^2\phi}{\partial x^2}+m'^2\phi=0$$

(Scalar Field)

[what physical system does this describe?

⇒ NL pendula + springs.

Calc'l plasma wave)

→ dispersion relation:

$$\omega^2=c_0^2k^2+m'^2$$

$$\Rightarrow \mathcal{L}=\frac{\dot{\phi}^2}{2c_0^2}-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2-\frac{m^2\phi^2}{2}$$

for nonlinearity:

$$U=\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2+\frac{m^2}{2}\phi^2+\frac{\alpha\phi^4}{4}$$

$$\Rightarrow \boxed{\frac{\partial^2\phi}{\partial t^2}-c_0^2\frac{\partial^2\phi}{\partial x^2}+m^2\phi=-\alpha\phi^3}$$

physics!

linear solution

$$\omega^2=c_0^2k^2+m^2$$

↳ unpert. speed

$$\phi=\phi(x-ct)=\phi(\theta)$$

↑ exact speed

→ look for wave train solutions!

↓

sneaky → converts to ODE problem.

so, for NL problem

$$\left[(c^2-c_0^2)\,\phi'' + m^2\phi = -\alpha\phi^3\right.$$

ϕ'

$$(c^2-c_0^2)\frac{\phi'^2}{2} + m^2\frac{\phi^2}{2} + \alpha\frac{\phi^4}{4} = \text{cst}$$

∴ expect nonlinearity will produce nonlinear phase velocity shift!

notation

$$\Rightarrow \left[\begin{array}{l} c = c^{(0)} + a^2 c_2 + \cdots \\ \phi = a\phi_1 + a^3\phi_3 + \cdots \end{array}\right\}$$ by correspondence with Duffing

n.b. here $\phi = \phi(\theta) \Leftrightarrow$ wave train solution
∴ only parameter is $\underline{\underline{c}}$.

$$\Rightarrow ((c^{(0)} + a^2c_2)^2 - c_0^2)\left[a\phi_1'' + a^3\phi_3''\right] + m^2\left[a\phi_1 + a^3\phi_3\right] = -\alpha\left[(a\phi_1) + a^3\phi_3\right]^3 = 0$$

$$(c^{(0)2} - c_0^2)\,\phi_1'' + m^2\,\phi_1 = 0 \qquad O(a)$$

$$(c^{(0)2} - c_0^2)\,\phi_3'' + m^2\,\phi_3 = -2c^{(0)}c_2\,\phi_1'' - \alpha\,\phi_1^3 \qquad O(a^3)$$

now, $O(a)$:

$$\phi_1 = \overset{1}{\cancel{\phi_0}} \cos k\theta$$ (contracted on a)

$$c^{(0)2} = c_0^2 + \frac{m^2}{k^2}.$$

~~$\omega^2 = \omega_0$~~

~~$c_0^2 k^2$~~

$\omega^2 = c_0^2 k^2 + m^2$

$O(a^3)$:

$$\left((c^{(0)})^2 - c_0^2\right)\phi_3'' + m^2\phi_3$$

$$= -2c^{(0)}c_2(\cos k\theta)''$$

$$-\alpha(\cos k\theta)^3$$

$$= 2c^{(0)}c_2k^2\cos k\theta$$

$$-\alpha\cos k\theta\left[\frac{1}{2} + \frac{1}{2}\cos 2\theta k\right]$$

$$= 2c^{(0)}c_2k^2\cos k\theta - \frac{\alpha}{2}\cos k\theta$$

$$+\frac{\alpha}{2}\,\frac{1}{2}\cos 3k\theta - \frac{\alpha}{2}\,\frac{1}{2}\cos k\theta$$

$$= 2c^{(0)}\overset{0}{c_2}k^2\cos k\theta - \frac{3}{4}\alpha\cos k\theta$$

$$+\frac{\alpha}{4}\cos 3k\theta$$

so, to kill secularity:

$$2c^{(0)} c_2 k^2 = \frac{3\alpha}{4}$$

$$c_2 = \frac{3\alpha}{8\underset{\uparrow}{c^{(0)}}k^2}$$

$$\therefore \quad c = c^{(0)} + a^2 c_2$$

$$= \left(c_0^2 + \frac{m^2}{k^2}\right)^{1/2}\left[1 + \frac{3\alpha a^2}{8k^2 c^{(0)2}}\right]$$

$$\boxed{c = \left(c_0^2 + \frac{m^2}{k^2}\right)^{1/2}\left[1 + \frac{3a^2\alpha}{8(c_0^2k^2 + m^2)}\right]}$$

nonlinear speed change / shift

recall, for linear forced SHO; (see 31)

$$a^2 = f^2/4m^2\omega_0^2\,(\epsilon^2+\lambda^2)$$ is {amplitude equation (trivial)

$\epsilon = \omega - \omega_{res}$; $\omega_{res} = \omega_0$

Then for NL system: (near primary, linear resonance) $\omega_{res} = \omega_0 + \kappa a^2$

$\kappa = \frac{3\,\cancel{\epsilon}\,\beta}{8\,\omega_0}$ ω_0 ↳ NL shift

→ defect

∴

$$a^2 = f^2/4m^2\omega_0^2\left((\omega-\omega_0-\kappa a^2)^2+\lambda^2\right)$$

$\epsilon^2 = \omega^2 - \omega_0^2$ ω_{res}

{ $a^2\left[(\epsilon-\kappa a^2)^2+\lambda^2\right] = f^2/4m^2\omega_0^2$

{ $a(\epsilon)$ relation ◁

cubic

⇒ cubic equation for a^2!! (3 roots → which?)

Contrast above

Observe:

- addition of NL ⇒ NL ω-shift ⇒ non-trivial amplitude equation

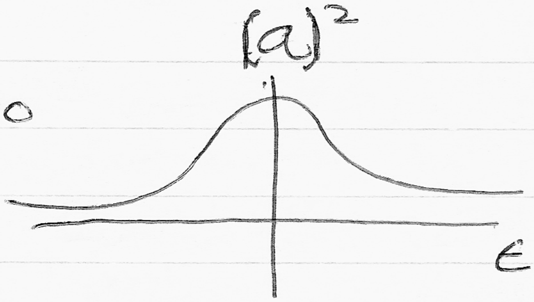
- for $\omega = \omega_{res}$ $(\epsilon = \kappa a^2)$, $a^2 = f^2/4m^2\omega_0^2\lambda^2$

∴ peak unchanged

but

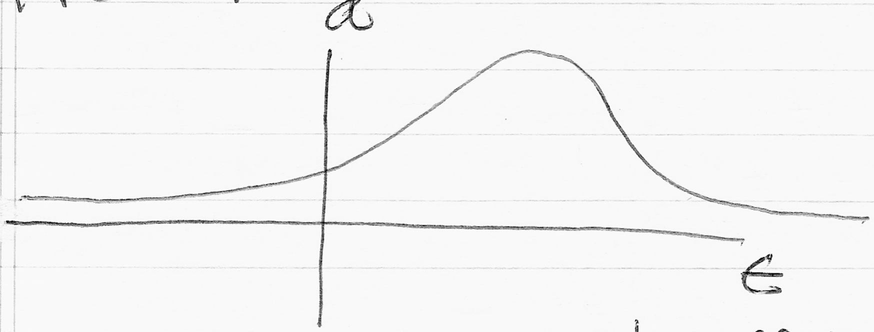
asymmetry induced in resonance curve!

i.e. $K = 0$



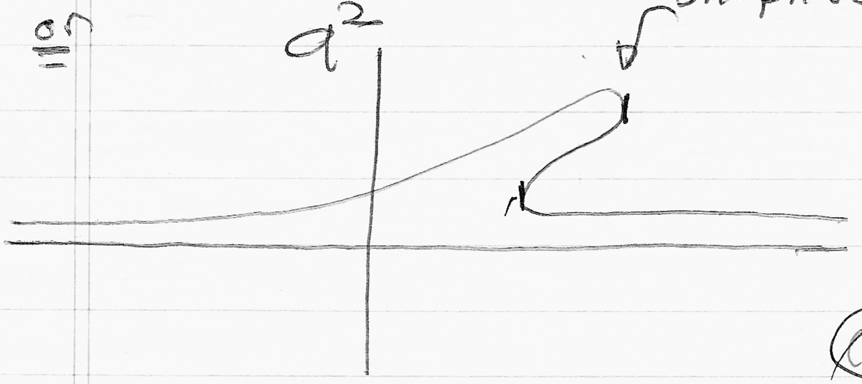
i.e. Lorentzian (symmetric)

$K \neq 0 \Rightarrow$



(asymmetric)

or



sharpness ~ $1/\lambda$ ⇒ NL more impt!

(double valued!)

(multi)

Can the resonance curve be double-valued?
→ yes! → when $d\bar{a}^2/d\epsilon \to \infty$.

Now, amplitude equation: $\bar{a}(\epsilon)$

$$a^2\left[(\epsilon - Ka^2)^2 + \lambda^2\right] = f^2/4m^2\omega_0^2$$

$$\Rightarrow$$

$$a^2\left[\epsilon^2 - 2K\epsilon a^2 + (Ka^2)^2 + \lambda^2\right] = f^2/4m^2\omega_0^2$$

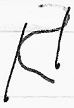
Can re-write amplitude equation as:

$$F(a^2, \epsilon) = a^2\left[\epsilon^2 - 2\epsilon(Ka^2) + (Ka^2)^2 + \lambda^2\right] - \frac{f^2}{4m^2\omega_0^2} = 0$$

so, for $da^2/d\epsilon$

$$\underset{\text{on curve (defn.)}}{dF = 0} = (\partial F/\partial a^2)\,da^2 + (\partial F/\partial \epsilon)\,d\epsilon$$

$$\Rightarrow \quad \frac{da^2}{d\epsilon} = \frac{-(\partial F/\partial \epsilon)}{(\partial F/\partial a^2)}$$



$\therefore$ $\underset{\text{(infinite slope)}}{\partial F/\partial a^2 = 0}$ for $\left\{\begin{array}{l}\text{(double valuedness)} \text{ blow up.} \\ \text{2 roots} \rightarrow \text{though } \partial F/\partial a^2 = 0 \text{ is coalescence pt. of two onset} \end{array}\right.$

$$F = K^2(a^2)^3 - 2\epsilon K(a^2)^2 + a^2(\epsilon^2 + \lambda^2) - \text{const}$$

$$\partial F/\partial a^2 = 3K^2(a^2)^2 - 4\epsilon K a^2 + (\epsilon^2 + \lambda^2) = 0$$

if $x = Ka^2$

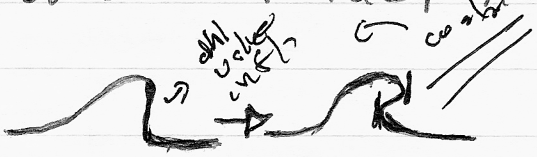
$$\partial F/\partial a^2 = 0 = 3x^2 - 4\epsilon x + (\epsilon^2 + \lambda^2)$$

$$\Rightarrow \quad x = \frac{4\epsilon}{6} \pm \frac{1}{6}\left(16\epsilon^2 - 12(\epsilon^2 + \lambda^2)\right)^{1/2}$$

$$= \frac{2}{3}\epsilon \pm \frac{1}{3}\left(\epsilon^2 - 3\lambda^2\right)^{1/2}$$

so $\kappa a^2 = \frac{2}{3}\epsilon \pm \frac{1}{3}(\epsilon^2 - 3\lambda^2)^{1/2}$ ⊛

∴ → double valued-ness at { inflection pt. / 2-root coalescence / Multi



i.e. when $\epsilon^2 = 3\lambda^2$, $\Rightarrow \kappa a^2 = 2\epsilon/3$

in terms external force magnitude ($a(\epsilon)$ relation)

$$f^2 = 4m^2\omega_0^2 a^2\left[\epsilon^2 - 2\epsilon\kappa a^2 + (\kappa a^2)^2 + \lambda^2\right]$$

$$\begin{cases} \kappa a^2 = \frac{2}{3}\epsilon = \frac{2\sqrt{3}}{3}\lambda & \text{(pk).} \\ \epsilon = \sqrt{3}\lambda & \text{(pk)} \end{cases}$$

and plugging into f^2 ⇒

$$\boxed{f_{crit}^2 = 32m^2\omega_0^2\lambda^3/3\sqrt{3}\,|\kappa|}$$

{ i.e. f > first required for inflection

→ for peak value of amplitude

max

$\partial F/\partial\epsilon = 0 \quad (\Leftrightarrow da^2/d\epsilon = 0)$

$\Rightarrow 2\epsilon - 2\kappa a^2 = 0 \Rightarrow \epsilon = \kappa a^2$

$\Leftrightarrow \omega - \omega_0 = \kappa a^2$, usual!

i.e. peak is at resonance (here $\omega = \omega_0 + Ka^2$), as usual.

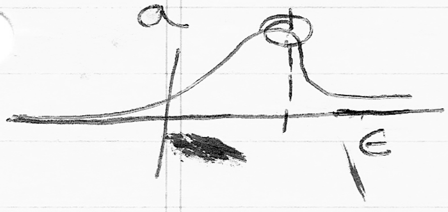
→ What's Going On?

- for $f^2 > f^2_{crit} = 32m^2\omega_0^3\lambda^3/3\sqrt{3}|K|$

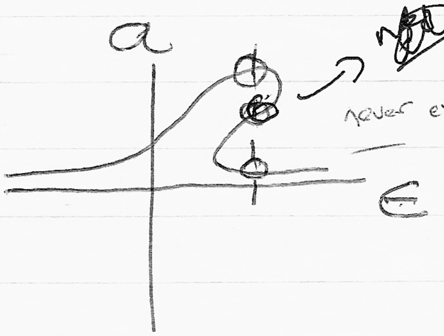
⇒ inflection occurs → bifurcation

⇓

⇒ 1 root → 3 roots



→



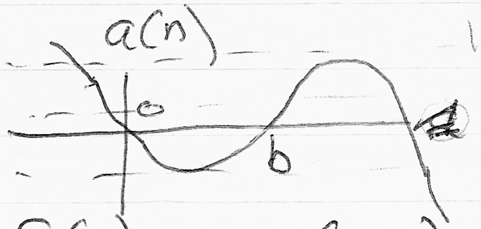
→ lower @ ~~[illegible]~~ → real. ε (no cntrpart in

never achieved

- of 3 roots ⇒ 2 stable
  1 unstable

i.e. demonstration

$$\frac{\partial n}{\partial t} = f(n)$$



$$f(n) = \underbrace{-n(b-n)(n-1)}_{a(n)} + S$$

$$\frac{\partial n}{\partial t} = 0 \Rightarrow S = +n(b-n)(n-1)$$

⇓

control ⇒ 3 or 1 roots, depending on S

but $n = n_{sol} + \delta n$

$$\frac{\partial \delta n}{\partial t} = \delta n \, f'(n_{sol})$$

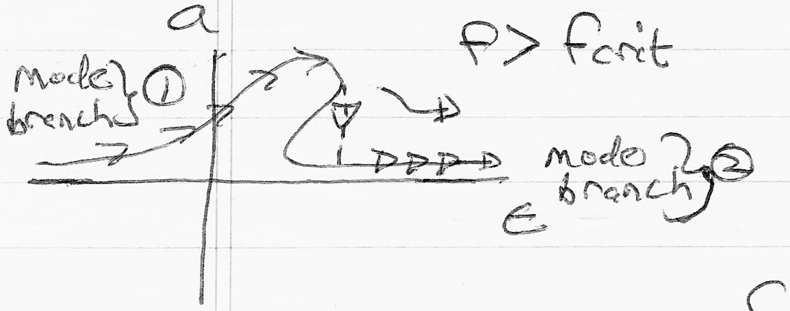
$$f' \gtrless 0 \rightarrow \begin{cases} \text{instability} \\ \text{stability} \end{cases}$$

i.e. $f' > 0$ → unstable root (1)

$f' < 0$ → stable roots (2)

Mode Tunneling

- 'bifurcation' occurs → jump between 2 stable branches (bifurcation ↔ inflection criterion)



$$\begin{cases} a(f, \epsilon) \rightarrow \text{surface} \\ \text{catastrophe} \end{cases}$$

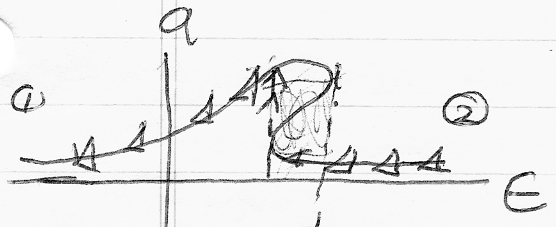
$\epsilon > \epsilon_{crit}$ ⇒ jump from $\begin{cases} \text{branch} \\ \text{mode ①} \end{cases}$ to $\begin{cases} \text{branch} \\ \text{mode ②} \end{cases}$

$f > f_{crit}$

- system exhibits "hysteresis"

i.e.

consider reversal of evolution from ② → ①, i.e.



$f > f_{cr.}$

↳ ϵ_{crit} for forward transition

to

ϵ_{crit} for back transition

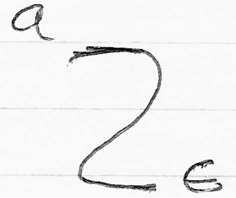
i.e.

$\epsilon_{fwd} > \epsilon_{back}$!

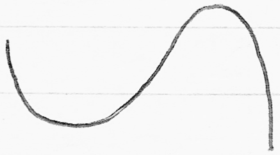
system tends to 'hang' in mode ②

→ driven Duffing oscillator is classic example of "S-curve" type bifurcation

i.e.



or

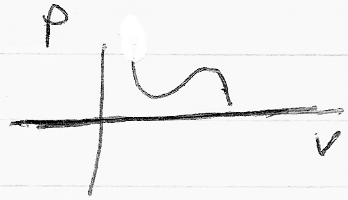


S curve ⇒ bi-stable system with unstable root in between, yielding transitions or mode-jumping

⇒ akin phase transition

①, ② ⟷ 2 phase

S curve ⟷ p V curve



mode jumping ⟷ first order transition

→ Now,

- have examined impact of NL on resonance phenomena, at primary/linear resonance
- but, is this the whole story?

⇒ NL-induced resonance phenomena?! → $\left\{\begin{array}{l}\text{new} \\ \text{resonance} \\ \text{physics}\end{array}\right.$

Now, re-insert αx^2 term:

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x + \underset{\substack{\uparrow \\ \text{quadratic} \\ \text{NL}}}{\alpha x^2} + \beta x^3 = \frac{f_{ext}}{m}$$

here $\omega_{res} = \omega_0 + \kappa a^2$

$$\kappa = \Big(\frac{3\beta}{8\omega_0} - \underset{\substack{\uparrow \\ \text{shift contribution,} \\ \text{due quadratic}}}{\frac{5\alpha^2}{12\omega_0^3}}\Big)$$

→ derive

why $O(\alpha^2)$?
→ $x^{(2)} x^{(1)}$ beat

Why α^2?

- observe $-\alpha (x^{(1)})^2 \rightarrow \frac{1}{2}\cos(2\omega_0 t)$ > non-resonant drives to $O(a^2)$

⇒

- lowest secular contribution from αx^2 nonlinearity is in $\underset{\substack{\downarrow \\ \text{here power } a\text{, not } \epsilon!}}{x^{(3)}}$ equation